Numerical Study of Boundary Layers With Reverse Wedge Flows Over a Semi-Infinite Flat Plate

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This paper presents the classical approximation scheme to investigate the velocity profile associated with the Falkner–Skan boundary-layer problem. Solution of the boundary-layer equation is obtained for a model problem in which the flow field contains a substantial region of strongly reversed flow. The problem investigates the flow of a viscous liquid past a semi-infinite flat plate against an adverse pressure gradient. Optimized results for the dimensionless velocity profiles of reverse wedge flow are presented graphically for different values of wedge angle parameter β taken from $0 \leq \beta \leq 2.5$. Weighted residual method (WRM) is used for determining the solution of nonlinear boundary-layer problem. Finally, for $\beta = 0$ the results of WRM are compared with the results of homotopy perturbation method.
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1 Introduction

There are many examples of fluid flows in technology and engineering where imposed boundary conditions results in flow separation. Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The analysis of boundary-layer problems has developed an interest in the field of aerodynamics. Also it has developed an interest in the analysis of a two-dimensional steady and incompressible laminar flow passing a wedge. Most nonlinear models of real-life problems are still very difficult to solve, either numerically or theoretically. Due to the chances of greater error in the approximate solution, numerical computation for high nonlinear models emerged as a powerful tool to get the results with greater accuracy. Flow separation is a phenomenon of widespread interest in boundary-layer theory. The boundary-layer problem of the flow past a semi-infinite flat plate against the pressure gradient was first studied by Falkner and Skan [1] in 1931 who considered an approximate solution of the equation. Accordingly, it is necessary to develop numerical methods capable of providing accurate solutions for these types of problems. In their pioneering work, Falkner and Skan [1] considered two-dimensional wedge flows (see Ref. [2]). They developed a similarity transformation method in which the partial differential boundary-layer equation was reduced to a nonlinear third-order ordinary differential equation, which could then be solved numerically. In 1979, Na [3] employed a group of transformations to reduce third-order boundary value problem to a pair of initial value problems and then solved these problems by means of a forward integration scheme. In 1983, Rajagopal et al. [4] studied the Falkner–Skan boundary-layer flow of a homogeneous incompressible second grade fluid past a wedge placed symmetrically with respect to the flow direction. In 1987, Lin and Lin [5] introduced a similarity solution method for the forced convection heat transfer from isothermal or uniform-flux surfaces to fluids of any Prandtl number. The solutions of the resulting similarity equations are given by the Runge–Kutta scheme. In 1997, Hsu et al. [6] studied the temperature and flow fields of the flow past a wedge by the series expansion method, the similarity transformation, the Runge–Kutta integration, and the shooting method. In 1998, Asaithambi [7] presented a finite-difference method for solving the Falkner–Skan equation. Later, Hsu and co-workers [6,8] presented a combination of a series expansion, similarity transformation, and finite-difference method for the heat transfer problem of a second-grade viscoelastic fluid past a plate fin. Mo et al. [9] used weighted residual method (WRM) to solve the Blasius equation and the Glauert-jet problem and produced highly accurate results. Kuo [10] presented the effect of heat transfer analysis by the differential transformation method for the Falkner–Skan wedge flow. He [11] introduced a new analytic technique homotopy perturbation method (HPM) to solve the nonlinear boundary-layer problem. In this present paper, the optimized results of boundary layers with reverse wedge flows over a semi-infinite flat plate were obtained by the WRM.The present study employs the WRM to obtain the solution of the Falkner–Skan boundary-layer problem. First, a group of transformations are used to reduce the third order nonlinear boundary value problem to a pair of initial value problems. These problems are then solved by the WRM and HPM. WRM provides a very powerful approximation solution procedure that is applicable to a wide variety of problems. WRM has the advantage that it solves the nonlinear differential equations directly, i.e., without the need for iterative calculations.

2 Problem Formulation

Consider the flow of an incompressible steady viscous fluid over a wedge, as shown in Fig. 1. It is assumed that the freestream velocity U is also uniform and constant, assuming that the flow is laminar and two-dimensional boundary layer.

A two-dimensional laminar viscous flow over a semi-infinite flat plate for the wedge flow is governed by

$$f'''(\eta) + \alpha_1 f(\eta) f''(\eta) + \alpha_2 - \alpha_3 [f'(\eta)]^2 = 0 \qquad (1)$$

with boundary conditions

$$\text{at} \quad \eta = 0: f(\eta) = 0, f'(\eta) = 0$$

$$\text{and as} \quad \eta \to +\infty: f'(\eta) = 1$$

where η is the similarity variable and is defined as

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and $f(\eta)$ is related to the stream function ψ by

$$f(\eta) = \frac{\psi}{\sqrt{\nu U_x}}$$

f is related to the u velocity by $f' = u/U$ and $f'(\eta)$ denotes the derivative of f with respect to η. When $\alpha_1 = -1$ and $\alpha_2 = \alpha_3 = -\beta$, Eq. (1) reduces to the boundary-layer outer reverse wedge flow.

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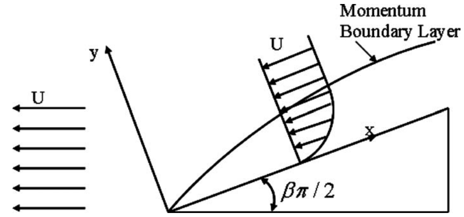

Fig. 1 Velocity profile for the Falkner–Skan reverse wedge flow

$$f'''(\eta) - f(\eta)f''(\eta) - \beta[1-(f'(\eta))^2] = 0, \quad \eta \in [0,+\infty] \quad (2)$$

with the new boundary conditions

$$\text{at} \quad \eta = 0: f(\eta) = 0, f'(\eta) = 0$$

$$\text{and as} \quad \eta \to +\infty: f'(\eta) = -1$$

Introducing a new constant m to replace β,

$$m = \frac{\beta}{2-\beta}$$

hence

$$\beta = \frac{2m}{m+1}$$

2.1 Method of Weighted Residuals. In this work, WRM is employed to get the numerical solution. This method comprises the following steps.

(1) Suggest a trial function or trial solution $f(\eta)$, which is at least twice differentiable.

$$f(\eta) = a + b\eta + c\eta^2 + d\eta^3 \quad (3)$$

$$f'(\eta) = b + 2c\eta + 3d\eta^2 \quad (4)$$

$$f''(\eta) = 2c + 6d\eta \quad (5)$$

(2) Use the boundary conditions. This yields the system of equations containing unknowns a, b, c, and d.
(3) Substitute the trial solution $f(\eta)$ in to the differential equation (2) to find the residual.
(4) Make the weighted residual zero on the defined domain by integral method, that is,

$$\int_0^\infty \text{residual} \quad d\eta = 0 \quad (6)$$

Solving Eqs. (3)–(6) simultaneously, we get the values of the constants a, b, c, and d, which gives the solution $f(\eta)$. This solution is given in Table 1 for different values of β.

Table 1 Values of $f(\eta)$ for positive values of β

β	$f(\eta)$
0.0	$-0.2337715508\eta^2 + 0.01783620677\eta^3$
0.5	$-0.2925848845\eta^2 + 0.02567798460\eta^3$
1.0	$-0.3140300741\eta^2 + 0.02853734321\eta^3$
1.5	$-0.3253249342\eta^2 + 0.03004332456\eta^3$
2.0	$-0.3323220702\eta^2 + 0.03097627603\eta^3$

3 Solution by HPM

Following the procedure developed by He [11] for the solution of nonlinear problems, the solution of Eq. (2) can be written as the power series in terms of an embedding parameter p as

$$f(\eta,p) = f_0 + pf_1 + p^2 f_2 + \cdots \quad (7)$$

In the limiting case when $p \to 1$, Eq. (7) yields

$$\lim_{p\to 1} f(\eta,p) = f(\eta) = f_0 + f_1 + f_2 + \cdots \quad (8)$$

where f_0, f_1, f_2, are the zero-, first-, and second-order solutions of Eq. (2), which can be determined by comparing the coefficients of $p^0, p^1, p^2, \ldots$ in the homotopic equation. The simplified form of homotopic equation is given by

$$H(f,p) = -f_2(\eta)\left[\frac{d^2}{d\eta^2}f_2(\eta)\right]p^5 + \left[-f_1(\eta)\frac{d^2}{d\eta^2}f_2(\eta) - f_2(\eta)\frac{d^2}{d\eta^2}f_1(\eta)\right]p^4 + \left[-f_0(\eta)\frac{d^2}{d\eta^2}f_2(\eta) - f_1(\eta)\frac{d^2}{d\eta^2}f_1(\eta) - f_2(\eta)\frac{d^2}{d\eta^2}f_0(\eta)\right]p^3 + \left[\frac{d^3}{d\eta^3}f_2(\eta) - f_0(\eta)\frac{d^2}{d\eta^2}f_1(\eta) - f_1(\eta)\frac{d^2f_0(\eta)}{d\eta^2}\right]p^2 + \left[\frac{d^3f_1(\eta)}{d\eta^3} - f_0(\eta)\frac{d^2f_0(\eta)}{d\eta^2}\right]p + \frac{d^3f_0(\eta)}{d\eta^3} = 0$$

comparing the coefficients of p^0, p^1, and p^2 on both sides to get the zero-order, first-order, and second-order solutions.

$$p^0: \frac{d^3}{d\eta^3}f_0(\eta) = 0$$

With $f_0'(\infty) = -1$, $f_0'(0) = 0$, $f_0(0) = 0$, we get the zero-order solution.

Table 2 Values of $f(\eta)$ for different values of β using WRM

η	$\beta=0$	$\beta=0.5$	$\beta=1$	$\beta=1.5$
0.0	0.0	0.0	0.0	0.0
0.5	−0.058648351667	−0.105552956102	−0.13358521896	−0.1543183171
1.0	−0.233012449954	−0.381091812141	−0.45922702758	−0.5121250414
1.5	−0.51507992018	−0.76856475046	−0.88732901993	−0.96201552953
2.0	−0.88687758167	−1.2200384352	−1.3619742154	−1.4469415571
2.5	−1.3225539306	−1.70239921161	−1.8544288256	−1.9431044837
3.0	−1.7957188324	−2.1970807138	−2.3525568342	−2.4422863324
3.5	−2.2865908977	−2.69576702085	−2.8521737115	−2.9421416204
4	−2.7841037125	−3.19550423184	−3.3521096729	−3.4421205838
4.5	−3.2835812597	−3.6954627531	−3.8521011113	−3.94211811409
5	−3.7835161376	−4.1954586184	−4.3521003612	−4.44211791864

Table 3 Values of $f'(\eta)$ for different values of β Using WRM

η	$\beta=0$	$\beta=0.5$	$\beta=1$	$\beta=1.5$
0.0	0.0	0.0	0.0	0.0
0.5	−0.23424998693	−0.40145519035	−0.49464930514	−0.55986314517
1.0	−0.46067606566	−0.68111550237	−0.77786529573	−0.83450720890
1.5	−0.66153361833	−0.85262208891	−0.91616827084	−0.94684424624
2	−0.81676344267	−0.94225051712	−0.97321681075	−0.98552669948
2.5	−0.91687846539	−0.98116712943	−0.99285127934	−0.99669373836
3.0	−0.96912318525	−0.99496107885	−0.99842430921	−0.99937287477
3.5	−0.99077499247	−0.99890652654	−0.99971608738	−0.99990216844
4.0	−0.99783495804	−0.99981048680	−0.99995871347	−0.9999875885
4.5	−0.99963914919	−0.99997574668	−0.99999546428	−0.9999987901
5.0	−0.9999999999	−1.0000000000	−1.0000000000	−1.0000000000

Table 4 Comparison of WRM with HPM for $\beta=0$

	$f(\eta)$		$f'(\eta)$		$f''(\eta)$	
η	HPM	WRM	HPM	WRM	HPM	WRM
0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	−0.0553683	−0.0586483	−0.2213007	−0.2342499	−0.440875	−0.4408754
1.0	−0.2319019	−0.23301244	−0.461721	−0.4606760	−0.4513362	−0.434417
1.5	−0.491665	−0.4916651	−0.6419180	−0.64191809	−0.3832223	−0.3832223
2.0	−0.8577638	−0.8577638	−0.8162297	−0.8162297	−0.383222	−0.3082851
2.5	−1.3002675	−1.3002675	−0.9449138	−0.9449138	−0.20209	−0.2020924
3.0	−1.7829608	−1.736690	−0.9948614	−0.9948614	−0.0964269	−0.0964269
3.5	−2.2784555	−2.2417597	−1.0066768	−1.0171510	−0.0152483	−0.0020813
4.0	−2.7858475	−2.6137836	−1.0020055	−1.0022166	0.010039	0.0476141
4.5	−3.2990751	−3.3741894	−0.9777101	−0.99909791	0.016069	0.0171657
5.0	−3.7907852	−3.8636006	−1.000000	−1.000000	−0.1402945	−0.1122776

$$f_0(\eta) = \frac{-1}{10}\eta^2$$

$$p^1: \frac{d^3}{d\eta^3} f_1(\eta) - \frac{1}{50}\eta^2 \qquad (9)$$

With $f_1'(\infty)=0,\ f_1'(0)=0,\ f_1(0)=0$, we get the first-order solution.

$$f_1(\eta) = \frac{1}{3000}\eta^5 - \frac{5}{48}\eta^2$$

$$p^2: \frac{d^3}{d\eta^3} f_2(\eta) + \frac{1}{10}\eta^2\left(\frac{1}{150}\eta^3 - \frac{5}{24}\right) + \frac{1}{15{,}000}\eta^5 - \frac{1}{48}\eta^2 \qquad (10)$$

With $f_2'(\infty)=0,\ f_2'(0)=0,\ f_2(0)=0$, we get the second-order solution.

$$f_2(\eta) = -\frac{11}{5{,}040{,}000}\eta^8 + \frac{1}{1440}\eta^5 - \frac{325}{4032}\eta^2 \qquad (11)$$

Putting Eqs. (9)–(11) in Eq. (8), we have

$$f(\eta) = -\frac{1}{10}\eta^2 + \left(\frac{1}{3000}\eta^5 - \frac{5}{48}\eta^2\right) + \left(-\frac{11}{5{,}040{,}000}\eta^8 + \frac{1}{1440}\eta^5 - \frac{325}{4032}\eta^2\right)$$

Simplifying, we get

$$f(\eta) = -0.2124131945\eta^2 + 0.0005736111111\eta^5 - 0.000001069444444\eta^8 \qquad (12)$$

Tables 2 and 3 show the current numerical results obtained by WRM for $f(\eta)$ and $f'(\eta)$ for positive values of β. Table 4 shows a good comparison of WRM with HPM for $\beta=0$. Unfortunately no data is available to compare other values of β.

4 Result and Discussion

The pressure gradient is the necessary condition for separation. The separation is caused by an adverse pressure gradient in boundary layer with negative velocity and positive pressure, which reverses flow close to the boundary. The results indicate that velocity profiles are associated with small values of the wedge angle parameter β taken from $0 \le \beta \le 2.5$. Figure 2 shows the dimensionless velocity profiles with different values of the parameters β. The wedge angle parameter is a measure of the pressure gradient. $f'(\eta)$ profiles merely squeeze closer and closer to the wall in a downward direction. The graph for $f'(\eta)$ gives asymptotic behavior, which converges to −1.

The dimensionless velocity distributions of the Falkner–Skan boundary-layer equation is plotted for various values of β. It is noted that for various positive values of β the difference occurs in the dimensionless velocity distributions. Figure 2 shows that $f'(\eta)$ decreases for positive values of β, which indicates a deaccelerating flow and for $\beta=0$ gives us a constant flow. Also for $\beta=0$, the

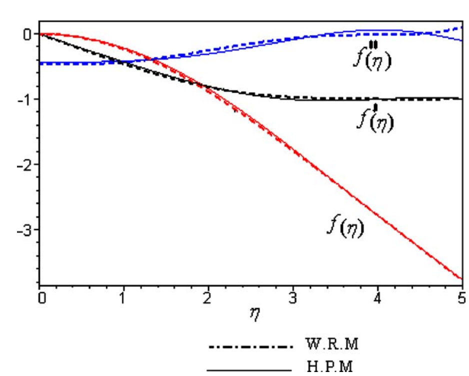

Fig. 2 Values of $f(\eta)$, $f'(\eta)$, and $f''(\eta)$ for different values of β

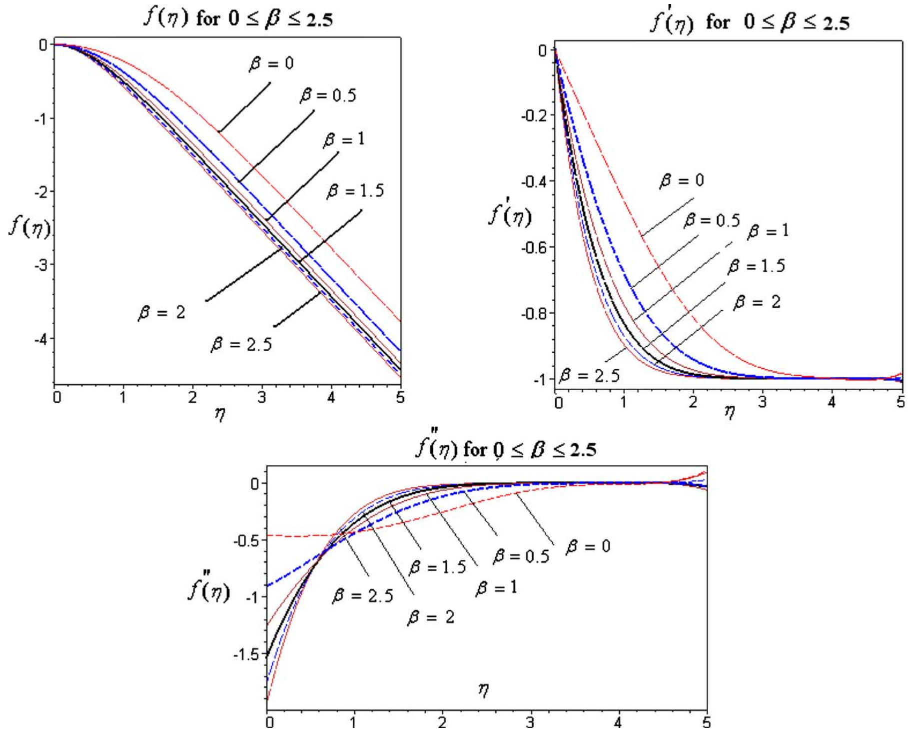

Fig. 3 Comparison of results obtained by WRM and HPM for $\beta=0$

results are compared with the homotopy perturbation method, which gives accurate results and for $\beta=0$ the graphs of $f(\eta)$, $f'(\eta)$, and $f''(\eta)$ were sketched in Fig. 3 using HPM and WRM, which shows that the flow is constant.

5 Conclusion

The present paper has discussed the applicability of the WRM to obtain the velocity distribution for a flow passing over a wedge. This numerical scheme, which has been applied within the current study, directly yields a close-form solution for a system of nonlinear differential equation. Numerical results of the Falkner–Skan boundary-layer problem has been presented in order to demonstrate the accuracy of WRM, which produced highly accurate results with negligible small error. In this paper, the proposed method provides an effective numerical scheme for determining the solutions of the nonlinear Falkner–Skan boundary-layer problem. Finally, the graphs have been sketched, which shows a behavior of flow in reversal for positive values of the wedge angle parameter β.

Nomenclature

α_j = material constants $(j=1,2,3)$
β = wedge angle parameter
η = similarity variable
ν = kinematic viscosity
p = parameter involved in HPM
ψ = stream function
x, y = Cartesian coordinate system
U = freestream velocity

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